ON THE MULTIPLE REALIZATIONS OF THE MAXIMUM SYSTEMIC PROBABILITY PRINCIPLE

by

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1970

I.) INTRODUCTION

In past work on **Relational Systems** we have separated and outlined the minimal essential components of a physical theory[1]. Namely, if in a prior decision system one is given only a delineation of the set of signs/symbols and experiential or formal relations thereon, but no particularized selection (initial values) thereof, the minimal composition of a spatial-temporal theory of the "real" includes the five classes of relations (A-E) of Appendix 1. On this basis, two classes of realizations of the ***maximum systemic probability principle (MSPP)*** - choose these subsystemic probabilities which maximize the systemic probability subject to all given relational constraints - were explicated[2], first as an "*initial probabilistic decision relation*" [1,3,4] (See Appendix 1; Section D) and second as a "*variational principle for generating subsystemic probabilistic dynamics*"[1,5,6] (See Appendix 1; section C:1:b:i:bb:2-2).

In the present note we extend this work to the consideration of the *resolution of probabilistic dynamics* through specific additional variational realizations of the MSPP (See Appendix 1; Section C:1:b:ii:bb). Although the major emphasis herein will be on the *variational analysis of stability under a perturbation*, the inclusion of a *variational analysis of stationarity* (See reference 5, Physical Review 124, 1682 (1961)) as a realization of the MSPP will be treated first since it is directly relevant to *stability analyses*. The secondary emphasis of this note will be on the conceptual unification provided by the MSPP in formalizing probabilistic mechanics. It should be obvious that the bulk of the work on which this suggested conceptual synthesis rests is scattered widely throughout the literature; however, the major contribution is unquestionably the work on the "path probability method"[5].

II.) PRELIMINARIES:

We take as given within a prior decision system the subsystemic probabilistic kinematics, experientially and formally (See Appendix 1; Section C:l:a), represented by classical probability theory[7] restricted by the Markovian assumptions (See Equation 2 below).

A.) Subsystemic Probabilities

Consider a system with a substructure described by the set of probabilities $\{p(i_t)\}$ where $p(i_t)$ is the probability for the single occurrence of the i^{th} subsystem at time t. The indices i = 1,2,...,n, label the particular subsystems. The interrelations in time among the various subsystems are given by the set of joint probabilities $\{p(i_t * j_{t'})\}$ where $p(i_t * j_{t'})$ is the probability of the joint occurrence of the subsystem i at time t and j at time t'. The conditions satisfied by the probabilities are:

(1) $$1 = \sum_i p(i_t) = \sum_{i,j} p(i_t * j_{t'}) = \sum_j p(j_{t'})$$

The conditional probabilities $p(j_{t'}/i_t)$, where $p(j_{t'}/i_t)$ is the probability that the j^{th} subsystem occurs at time t' conditional upon the occurrence of the i^{th} subsystem at t, are related to the single and joint subsystemic probabilities through the equation

(2) $p(i_t \ast j_{t'}) = p(i_t)p(j_{t'}/i_t)$

These conditional probabilities become physically more descriptive when interpreted as transition probabilities.

B.) Systemic Probabilities

We now define the following general systemic probability functions.

(Definition 1) $P^1(t) = P^1$ [single subsystemic probabilities at time point t]

(Definition 2) $P^2(t;t') = P^2$[single, joint or conditional subsystemic probabilities interrelating the time points t and t']

Additionally, we shall refer to the function

(Definition 3) $S^1 = k \ln P^1$

as a generalized one (time) point entropy, and to

(Definition 4) $S^2 = k \ln P^2$

as a generalized multiple (e.g. two, and hence infinitely many, time) point entropy. For examples of P^1 and P^2 see reference 5, Math. Anal. Appl. 4, 488 (1962); for S^1 and S^2 see ref. 6. For certain forms of P^1 and P^2, the equation

(3) $S = k \ln P$

in conjunction with Sterlings' approximation reduces to the familiar Shannon information (entropy) function[(8)]. Since equation (3) can be solved for P as

(4) $P = e^{S/k}$,

the kinematical relations for S imply a kinematics for P. Hence classical information theory[(7)] can, at least partially, be taken as the systemic probabilistic kinematics (See Appendix 1; C:2:a:i).

III.) ON THE GENERAL FORMS OF THE REALIZATIONS OF THE MSPP

A.) One Point Variations

The general form of the MSPP in this case is as follows:

(MSPP1) At a specific time point t the optimal choice of the $\{p(i_t)\}$, realizing certain relational constraints, is given by those $\{p^*(i_t)\}$ which maximize P^1 (or S^1).

(Proposition 1) With S^1 as the classical information function, MSPP1 becomes the well-known maximum entropy principle[1, 2, 3, 4].

B.) Multi-(two) Point Variations

In this case there are several different forms of the MSPP depending explicitly upon what subsystemic probabilities are varied and which are held fixed. The following paraphrases of specific forms of the principle, the first two as applied by Kikuchi[5], and the third one by ourselves[6], demonstrate the point for cases in which $\tau = t' - t$ is a small time interval.

(MSPP2n) The "natural" (optimal) path which a system takes during τ is derived by maximizing $P^2 (t; t') = P^2(\{p(i_t \cap j_{t'})\})$ holding the initial $\{p(i_t)\}$ fixed.

(Proposition 2) In terms of the $\{p^*(i_t \cap j_{t'})\}$ which maximize P^2, together with the consistency relations (equation 1) and a limit relation, MSPP2n is in the path probability variational procedure for generating the "natural" dynamical relations on the $\{p(i_t)\}$[5].

(MSPP2f) The fluctuating path connecting $\{p(i_t)\}$ and $\{p(j_{t+\tau})\}$ is obtained by maximizing P^2 with respect to $\{p(i_t \cap j_{t+\tau})\}$ holding the initial $\{p(i_t)\}$ and final $\{p(j_{t+\tau})\}$ fixed.

(MSPP2o) The optimal probabilistic dynamical relations on the $\{p(i_t)\}$ are derived from the $\{p^*(i_t \cap j_{t'})\}$ which maximize $S^2 (t;t') = S^2(\{p(i_t \cap j_{t'})\}; \{p(j_{t'}/i_t)\})$ subject to any given relational constraints including the fixation of the $\{p(j_{t'}/i_t)\}$.

(Proposition 3) When the dynamical relations on the $\{p(i_t)\}$ are obtained from the $\{p^*(i_t \cap j_{t'})\}$ through the consistency relations and a limit process, MSPP2o is recognized as the optimal entropy variational principle[6].

The general use of two point variational forms of the MSPP for generating probabilistic dynamics has been discussed previously[1,2]. In the next section we turn to the major point of this note, i.e., the elucidation of the MSPP as a variational process for resolving these dynamics.

IV.) SPECIFIC REALIZATIONS OF THE TWO POINT VARIATIONAL FORM OF THE MSPP IN RESOLVING PROBABILISTIC DYNAMICS

A.) Determination of the Stationary Subsystem

Let $\vec{p}^{(\nu)}(t)$ stand for $\{p(i_{t+\nu\tau})\}$ and consider the sequence $\vec{p}^{(0)}(t)$, $p^{(1)}(t+\tau)$, $\vec{p}^{(2)}(t+2\tau)$, … … … The systemic probability that the sequence of subsystems occurs is represented by

(5) $P^{\nu\tau} = P^{\nu\tau}\,[p^{(0)}(t); p^{(1)}(t+\tau); p^{(2)}(t+2\tau); \ldots] = \Pi\, P^2\,[p^{(\nu)}(t+\nu\tau)/p^{(\nu+1)}(t+(\nu+1)\tau)]$
$\nu=0, 1, 2,\ldots$

The probability that this sequence be optimal in the sense that P^2 is a maximum is given by

(6) $P^{\nu\tau} = P^{\nu\tau}_m = \Pi\, P^2_m\,[\vec{p}^{(\nu)}_m(t+\nu\tau); \vec{p}^{(\nu+1)}_m(t+(\nu+1)\tau)]$

where P^2_m is the maximum of P^2 as generated according to MSPP2n, MSPP2f, etc.

Now suppose $\exists$ N s.t. $\nu > N$ (i.e., $\nu = N+1, N+2, \ldots$) $\Rightarrow \vec{p}^{(\nu)}(t+\nu\tau) = \vec{p}^{(N)}(t+N\tau)$ and
$\nu < N \Rightarrow p^{(\nu)}(\vec{t})$ □ $p^{(\nu+1)}(t+(\nu+1)\tau)$.

(Definition 5) $\{p(i_{t+N\tau})\}$ corresponds to the (unique) stationary subsystem in the sequence. If for whatever initial t the same subsystem is stationary in the respective sequence, then that subsystem is (uniquely) stationary in the system.

As a consequence of these definitions, we have the following variational principle:

(MSPP2s.s.) For systems evolving optimally to a unique stationary subsystem $p_{s.s.}$, the latter exists as the final $p^{(N)}(t+N\tau)$ for sequences covering a "sufficiently long" time interval $(t \rightarrow t+N\tau)$ and for which $P^{N\tau}$ is a maximum.

(Proposition 4) If, for the "sufficiently long" time interval N , $P^{N\tau}$ is approximated as $P^{N\tau} \cong (P^2[\vec{p}_{s.s.};\vec{p}_{s.s.}])^N$ and $P^{N\tau}$ is maximized via P^2 according to MSPP2f, then, for the "natural" (optimal) evolution of the system, MSPP2s.s., becomes the "maximum persistency" principle (See ref. 5. Phys. Rev. 124, 1682 (1961).)

In an analysis of Klein's (9) two-level atom problem, Kikuchi (ref. 5., ibid.) has shown that the maximum persistency principle leads to the same stationary subsystem as derived from the kinetic equations and that the principle applies even when Prigogine's minimum entropy production principle (10) fails. The dependency of one method of analysis for stability upon the entropy production will he noted in the following section.

B.) Analysis of Stability

(Definition 6) A system, realized through the occurrence of a given subsystem, is stable (in some degree) under a perturbation if the subsystem recurs. otherwise it is unstable. Absolute stability <=> certain recurrence. Conversely, absolute instability <=> impossible recurrence. Finally, potential stability (stabilizability) <=> a (non-zero) probable recurrence, and the probability of recurrence essentially measures the degree of stabilizability.

These definitions in conjunction with equation 5 imply Proposition 5 below as a formal proposition on stability. Let $\vec{p}^{(0)}(t) = p_{p.s.}$ correspond to the perturbed subsystem (p.s.). If $\exists$ a sequence $\vec{p}^{(0)}(t); \vec{p}^{(1)}(t+\tau);\ldots; \vec{p}^{(N)}(t+_N\tau)$, however large, starting from p.s. and terminating with the recurrence of the given unperturbed subsystem (u.s.) with corresponding $\vec{p}^{(N)}(t+_N\tau) = \vec{p}_{(u.s.)}$ then we have the following proposition.

(Proposition 5) The system (realized through the occurrence of the u.s.) is potentially stable, i.e., stabilizable, iff $P^{N\tau} > 0$. $P^{N\tau} = 0$ for all sequences terminating with $\vec{p}_{u.s.}$ <=> the system is absolutely unstable. Absolute stability on the other hand <=> $P^{N\tau} = 1$.

Since the demonstration of potential stability depends only on finding one sequence s.t. $P^{N\tau} > 0$, we have the following variational principle.

(MSPP2S-I) A system is potentially stable iff the maximum $P_m^{N\tau}$ of $P^{N\tau}$, for a "sufficiently long" interval N and with the initial $\vec{p}_{s.s.}$ and final $\vec{p}_{u.s.}$ fixed, is positive definite. $P_m^{N\tau}$ <=> system is absolutely unstable.

To demonstrate the relationship of the above to existing methods of analyzing for stability, we first of all observe the relation between this last variational principle (MSPP2S-I) and the maximum persistency principle (Proposition 4.). In particular, if $\vec{p}_{u.s.} = \vec{p}_{s.s.}$ and $P^{N\tau}$ is given as in Proposition 4, then the maximum persistency principle, as a special case of MSPP2s.s., is implied by MSPP2S-I.

1.) Pseudo-Entropy (See ref. 5, <u>Phys. Rev. 124</u>, 1691 (1961)) Defining a quantity G as

$$(7) \qquad G = G[\{p(i_{t=-\infty})\};\{p(i_t)\}] = P2\ (-\infty; t) = P2\ [\vec{p}_{s.s.} = \{p(i_{t=-\infty})\};\{p(i_t)\}],$$

The psuedo-entropy is given by

$$(8) \qquad \tilde{S} = \tilde{S}[\{p(i_{t=-\infty})\}; \{p(i_t)\}] = S^2\ (-\infty; t) = k \ln G$$

In terms of these latter we have the following theorems:

(Proposition 6) The pseudo-entropy is a maximum for the stationary subsystem.

(Proposition 7) The rate of pseudo-entropy production vanishes for the stationary subsystem, and is positive otherwise.

It has further been shown that there exists a specific (conditional) functional relation between the pseudoentropy production and the persistency. In view of the implication of the maximum persistency principle in the MSPP2S-I, it follows that there should exist a relation between this latter and the entropy production in the analysis of stability. It is as follows:

2.) Entropy Production in the Analysis of Stability

The following proposition has been shown to represent a necessary (but not sufficient) condition for absolute stability.

(Proposition 8) For all perturbations, a stationary sub system is absolutely stable if the variation in entropy production is positive and unstable if it is negative definite.

3.) Entropy in Liapunov's Direct Method[12]

It has been shown for N chemical species in a homogeneous chemical reaction system in which the rates of change of the concentrations $\{C(i_t)\}$ are given by massaction rate equations for the elementary single-step processes with true equilibria as the final subsystemic realizations of the system, that

$$\text{(Proposition 9)} \quad H \equiv \sum_i [C(i_t) \ln C(i_t)/\bar{C}_{(i)} - C(i_t) + \bar{C}_{(i)}]$$

is a Liapunov function for the system.

For the many-level system considered by Kikuchi (See ref. 5, Phys. Rev. 124, 1691 (1961)) G can be written as

$$(9) \qquad G = \exp [\, N \sum_i P(i_t) \ln \frac{p(i_{t=-\infty})}{p(i_t)} \,]$$

Hence,

$$(10) \qquad \tilde{S} = RN \sum_i p(i_t) \ln \frac{p(i_{t=-\infty})}{p(i_t)}$$

With $Np(i_t)$ and $Np(i_{t=-\square}) = \bar{C}_i$, we have

$$(11) \qquad H = \frac{1}{K}\, \tilde{S} - \sum_i (C(i_t) - \bar{C}_{(i)})$$

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APPENDIX 1.

A.) Order (Deterministic) Relations
    1.) Deterministic Kinematics
    2.) Deterministic Dynamics
B.) Initial Order (Deterministic) Decision Relations Taken as arbitrary choices of the initial spatial-temporal relations.
C.) Order-Disorder (Probabilistic) Relations
    1.) Subsystemic Probabilities
        a.) Probabilistic Kinematics
        b.) Probabilistic Dynamics
            i.) Construction
                aa.) Experiential Construction
                bb.) Formal Construction:
                    1-1.) Deterministic Extension:
                        Via the imposion of the deterministic dynamics on the probabilistic Kinematics
                    2-2.) Variational Construction:
                        aaa.) Direct: Maximization of the systemic probability (over a time interval) with respect to the appropriate subsystemic probabilities.
                        bbb.) Indirect: Maximization of the systemic probability (over a time interval) via the extremization of a subsumed "Lagrangian" wrt the appropriate subsystemic probabilities.
            ii.) Resolution
                aa.) Direct:
                bb.) Variational Procedures: Use of the MSPP principle in either exact (analytic) or approximate, complete or partial, resolution processes, e.g., in the analysis of
                    1-1.) Stationarity, and
                    2-2.) Stability under a perturbation.
    2.) Systemic Probabilities
        a.) Probabilistic Kinematics
            i.) information (Entropy) Theory
        b.) Probabilistic Dynamics
            ii.) Time Evolution of the Entropy
D.) Initial Order-Disorder (Probabilistic) Decision Relations
    Maximization of the systemic probability (at a selected time point) with respect to the appropriate subsystemic probabilities
        (Note: All extremizations are to be taken as experientially conditioned (constrained)).
E.) Action (Relation-to-the-"real")