SYMBOLIC RELATIONAL SYSTEMS:

THE MULTILEVEL APPROACH TO THE CENTRAL NERVOUS SYSTEM*

J. R. Hamann and L. M. Bianchi

*This manuscript is an extended version of a paper written during the 1968 Summer Colloquium on Theoretical Biology, at Traverse City, Michigan, sponsored by NASA and directed by Harold Morowitz. That paper was published as "Neural Nets as Relational Systems" in the Quarterly Bulletin from the Center for Theoretical Biology, 1, No. 2, 81 (Sept. 1968).

State University of New York at Buffalo
Faculty of Natural Sciences and Mathematics
and
Center for Theoretical Biology
September, 1968
Revised, January, 1969

I. INTRODUCTION

Most theories concerned with the understanding of the functioning of the central nervous system fall into two broad classes, corresponding to two clearly distinct levels of biological organization - the cell and the (sub)organ. In the latter approach the brain is viewed as an aggregate of co-operative and competing organs. These units are defined by and identified with their histological and biochemical differentiation and their specialized function. The importance of this way of proceeding stems from the fact that these functions are actually expressible in terms of psychological and macrophysiological behavior and are thus, at least to a certain extent, directly observable. It is clear, however, that to understand many of the detailed properties of global behavior it is necessary to know the internal structure and operation of the black boxes in terms of which this block diagram of the brain is constructed.

We have then the other class of theories which consider the neuron as the basic unit. Incidentally, this does not mean that we exclude the existence of intermediate organizational levels. In fact, they have been demonstrated-receptive fields at various stages along the sensory pathways, layered and columnar structures in the cortex, etc. The point is simply that we probably gain in going to the elementary unit underlaying <u>all</u> subsequent levels, since very often these intermediate structures are identifiable only in terms of inputs to single neurons (e.g. the receptive field of a retinal ganglion cell).

All neuronal theories obviously face the question of how to explain the global behavior of the brain or of its suborgans as the resultant of the behavior of single neurons plus their mutual interactions. Obvious as it may seem (methodologically) the approach to this problem has frequently been obscured by some confusion in the definition of such concepts as determinism and stochasticity (order and disorder). In an ordered (deterministic) system one knows explicitly which propositions on the system are true. In a disordered (random) system one knows only that there is at least one true proposition in the ordered set, but one does not know which specific proposition it is. One can only assign probabilities to the propositions measuring their "truth" relative to given evidence. The ordered (deterministic) system on this basis appears strictly as the special instance of the order-disorder structure when one probability is unity and all others are zero. It is important to notice that frequently it is useful to construct a deterministic system from a stochastic one by constructing propositions on expected values in the latter such that the probabilities again become 0 or 1.

It should be noted that the concepts of order and disorder can only be understood together. That is, to say a system is ordered implies we have previously decided what is disordered and conversely. Consider for example, the case of the (sequential) system; abc. This sequence subsumes the (subsequential) subsystems ab and bc. The sequence cab, in contrast, subsumes the subsequences ca and ab. We then say that the system abc is an "alphabetical" order system whereas cab, relative to the former, is alphabetically disordered since it does not subsume the same subsequences. The essential connection between order and disorder is made clear when we further observe that cab is a "word order" system and, relative to this, abc is disordered.

A further blurring of the issue arises from the seemingly diffuse failure to recognize the need of a hierarchy of inferential methods in going from the neuronal level to that of the organ. In this context there are essentially only two possibilities - the dynamical equations that describe the behavior in time of the single neuron (and of any system for that matter) can be either deterministic or stochastic. Suppose first that the neuron is a stochastic system[(1)] (e.g. even though it is not in a refractory state and the driving current exceeds the threshold, it can either fire or not fire, with certain probabilities). To solve the set of equations describing a network of neurons at a certain time t, we need to know an initial state of the system, that is the prior probability distribution functions for all the variables entering our equations. If we consider now the deterministic case, we are faced with a similar (though, a priori, not necessarily equivalent) problem to find the initial values of all the variables. This is of course operationally impossible, due to the size of our system. All we can do is to define a set of such systems, each one with a different initial state compatible with the available dynamics. The problem is then to weigh these various possibilities, in other words to infer a prior probability distribution function. The important point is that if we just make use of the dynamical equations, all we can say about these probabilities is that they be all equal. This fact simply reflects the intrinsic impracticability of predicting the behavior of a system purely in terms of the behavior of the large number of its elementary subsystems. In other words, we have to explicitly take into account whatever knowledge we have of the system as peculiar to the system as a whole (e.g., conserved quantities, expected values of observables at a given time, etc.). We can then use this information as a constraint in determining the initial distribution functions in an optimal way. That there exists such a way is demonstrated in Sec. II, where the Principle of Optimal Entropy Inference is discussed. Probably the most powerful feature of this principle is its ability in providing us with a tool for treating the time dependent aspects of the statistical dynamics of neural nets, namely their behavior far from equilibrium, in fact far from stationary states. Indeed it is not very surprising in view of the origin of classical statistical mechanics, that all attempts at applying the methods of statistical mechanics to neural nets had to limit themselves to equilibrium states, despite the fact that the most interesting aspects of the behavior of neural nets are time dependent (adaptation, facilitation, learning, etc.), and very often presumably irreversible.

What about the concrete possibility of carrying out the program we have sketched above? We will try to suggest some answers in Sec. III. Certainly a difficult point is that of defining appropriate macroscopic observables for which expected values can be experimentally determined at a given instant in time.

If we accept the current trend in physiological psychology, according to which essentially all psychological functions exhibit a bioelectric counterpart (observable in the various EEG, LEEG, ECG, etc.), then the obvious places where we have to look for macroscopic observables are the recording of gross evoked and maintained potentials. One important possibility is that some aspects of the distribution function be *stable*, in the sense that the convolution of any number of such functions be again of the same type. In this case one could try to identify such properties directly in the electrograms. An example of this sort is proposed by J. Cowan. *(2)

Since the stochastic theory of neural nets, as we are going to propose it, is a particular realization of relational systems and since we want to stress the conceptual unification that the latter brings about in a vast number of areas of specialized knowledge, we will outline in some detail the general structural aspects of relational systems (Sec. II).

II. RELATIONAL SYSTEMS: GENERAL ASPECTS (3)

A. Preliminaries

The fundamental or presupposed notions upon which this formulation rests, and by the realization of which it is formed, are those of "system" (including specifically subsystem and image system), "relation"** and a special form of a relational system which we call a "decision system". This latter, when hierarchically repeated, will form the principle unit in our construction. The specialized form of this elementary structure is called a formal decision system, is represented as

$$S \overset{R}{\underset{D}{\longleftrightarrow}} S',$$

and is a particular instance of what we refer to as a formal symbolic system. S and S' are symbols representing two systems (or sub-systems), R represents the relation between S and S', and D symbolizes the decision relation. In addition to the above, the derived notions of order, disorder, and consistency, as well as the real, possible, and probable will be employed. The concepts of order (and disorder) and consistency are implied by the relation of subsumption (and its negation). The real and possible are given in terms of the notion of consistency (with experience) whereas the probable is in turn explicated on the basis of the former pair.

B. Relational Systems Decision Hierarchy

1. Presumed Decision Hierarchy

Within a presupposed decision system, all theories are structured as decision hierarchies. The problem of construction here is to generate the minimal hierarchy of decision units which constitutes a complete physical stochastic relational system. Deterministic relational systems are contained therein when the underlying order-disorder relations degenerate to pure order relations.

2. Order Relations

Experience, imaged as a formal symbolic system can always be related to-and-in language and hence represented, for example, as a propositional system (PS), a proposition being a statement which is either true or false. "Ordinary" experience when imaged by a system of propositions can be ordered according to a set of relations composing a Boolean algebra (BA). [(4)] Hence the initial structure in our decision hierarchy is the Boolean algebra,

$$PS \overset{R}{\underset{D}{\longleftrightarrow}} sPS,$$

where sPS is the theorem set in the algebra deriving from the propositional system PS (i.e., the set of elementary symbols together with the experientially*** presumed relations, or axioms) via the relational forms R, and decision rules, D. Since we need only know the system PS explicitly to proceed to the next level in the hierarchy, R, D and sPS

---

* Since this paper is essentially methodological, we will not be directly concerned in a critical examination of the various types of dynamics that have been proposed to describe the neuron. So we will arbitrarily take Cowan's model as our starting point, the choice being dictated essentially by his development of a statistical neuromechanics based on his dynamics. We then hope to show the decisive advantages of our more general approach by enabling that theory to cover the wider domain outside equilibrium states.

** Any system is necessarily relational, if one includes the possibility of self-reference.

***"Experiential" is the adjective corresponding to the noun "experience", and is considered more inclusive than, e.g., ."experimental".

will not be discussed further. The relations of PS which are required, however, are given explicitly as the axioms of the BA (see ref. 4, p.10). It should also be noted that this structural level, via sequential adjunctions of specified R and D, subsumes the various forms of deductive logic.

Although the order structure for "ordinary" experience is the BA, this need not be true in general for symbolically imaging the real. Indeed, it is known not to be true for quantum systems.[5] We will return to this point again in Section III.

3. Order-Disorder Relations

a. Subsystemic Probability Relations

i. Classical Probabilities

We begin by defining probability[3] as a relation between the real and the possible in becoming, i.e., in the actualization of the real. This is symbolized as $p(a_i/\eta)$ where p is the probability that the proposition a_i is true given the hypothesis η. a_i represents the possible, that is, a proposition in a set of N propositions, i=l,...,N, at least one of which is true, but specifically which one(s) being unknown. η images the real (experience, evidence).

It has been shown[4] that two simple experiential assumptions representing, e.g., our statistical experience in coin-tossing, together with a restricted set of formal assumptions, serve to uniquely generate classical probability theory when the propositions are required to satisfy a BA.

ii. Generalized Probability Theory[3]

Given the BA and the experiential and formal assumptions referred to above, classical probability theory results. If, however, one changes, in part, the underlying order structure or the probabilistic assumptions one has the potential relational structure for the construction of generalized probabilities. In the case of quantum systems, this is the procedure we are following to generate an axiomization of quantum probability theory. Again, we will return to this point in Section III.

b. Systemic Probability Relations [3]

In the previous case $p(a_i/\eta)$ was a relation between the individual propositions (or their conjunctions or disjunctions) in the set $\{a_i\}_N$ and the hypothesis η. This relation we termed a subsystemic probability. It is also meaningful, however, to construct a relation between η and the set $\{a_i\}$ considered as a single system. This we call a systemic probability relation. Since this latter includes the information (or entropy) function as a special case, and in that information theory is now widely applied, we confine our attention in this section to an outline of the details of the entropy function.

i. Information Theory

(a.) Classical probabilistic systems [4]

Given an exhaustive system of m propositions on the evidence η, three experiential assumptions on the entropy, S, representing a formal codification of our knowledge of certain characteristics of rational choice, together with certain formal assumptions yield the most general, uniquely determined form of S as

$$S[b_1,b_2,\ldots,b_m/\eta] = -K[\sum_i p_i \, ln p_i + \sum_{i<j} p_{ij} \, ln p_{ij} - \ldots \pm p_{1\ldots m} \, ln p_{1\ldots m}]$$

(where $p_i = p(b_i/\eta)$, $p_{ij} = p(b_i \bullet b_j/\eta)$, etc.)

If the propositions in the set $\{b_i\}_m$ are mutually exclusive as well as exhaustive, only the first sum on the r.h.s. of the equation for S remains. This corresponds to the usual formula for the entropy.

At this point we have completed the kinematic aspects of the order and order-disorder levels in the decision hierarchy. Within this level there is subsumed a system. of stochastic evolutionary equations generally representable as

$$F_t = K_{t,t_o} F_{t_o}, \quad \text{or}$$

stochastic inferential equations of the form

$$F_{posterior} = F^{prior}_{posterior} \, F_{prior}$$

That is, the relational system tells us how to obtain F_t (some probability function at time t) if the function is known at some prior time t_o. Or it generates a posteriori functions given a priori functions. There is nothing in the system, however, to tell us how to construct the unknown probabilities F_t or F_{prior}. To accomplish this and thus generate a complete stochastic relational system, we must admit a further decision structure.

4. Optimal Probabilistic Inference

a. Optimal Entropy Inference (OEI) [6-9]

Writing

$$P = \Pi_i p_i^{-p_i},$$

we obtain

$$S = K \, ln \, P$$

for the exhaustive, mutually exclusive system of propositions. P here represents a specific, approximate instance of what we mean by a systemic probability. If the subsystemic probabilities p_i are unknown, their most plausible choice would be that which makes the system the most probable, i.e., that which maximizes P. Since a maximum in P implies a maximum in *ln*P, or in K *ln* P, with K a constant, this also implies a maximum in S. This is codified in the following decision rule called the *optimal entropy principle; an optimal choice of the unknown subsystemic probabilities is achieved by extremizing S subject to* $\mathbf{h}'$*(*$\mathbf{h}'$ *is evidence equal to, subsumed by, or in addition to* $\mathbf{h}$*).*

In terms of these first three levels, namely the order, order-disorder, and optimal probabilistic inferential structures, we have a formal stochastic relational system which needs only to be dynamically completed. To achieve a meaningful specialization of the stochastic dynamics, an additional level in the decision hierarchy is required based on the adjunction of experiential (e.g., space-time) relations among the system variables.

5. Experiential Systemic-Subsystemic Relations

a. Space-Time Relations

Given a system specified by propositions on the set of space-time variables (Z,t)*, the general stochastic dynamics can be specialized by constraining it to the experiential functional time-dependence of the space variables. The following forms of these space-time relations are possible.

i. Ordered (Deterministic) Space-Time Relations

In this case the space-time relations are taken to be ordered (non-random) functions of (Z,t). One then has the following instances of the initial conditions, (Z_o,t_o) essential for solving the space-time equations.

(a.) Ordered (Deterministic) System Variables

In this case one can assert with certainty which proposition on (Z_o , t_o) is true, i.e. one knows Z_o at t_o with a probability of unity. Hence the disorder level vanishes from our hierarchy.

(b.) Disordered (Random) System Variables

Under this condition, although the experiential space-time equations are strictly deterministic, one can only assign probabilities to a set of propositions on the possible initial values of Z_o at t_o Hence the dynamical problem to be solved is really the stochastic one generated by constraining the general stochastic dynamics to be consistent with this deterministic dynamics. The problem of choosing the initial probabilities is the essence of classical equilibrium statistical mechanics. As we have already noted, this decision problem is resolved by the method of optimal entropy inference, which has in turn been shown to conceptually subsume all that is known in statistical mechanics and thermodynamics, both reversible and irreversible.

ii. Disordered (Random) Space-Time Relations

The disordered (or random) space-time relations are characterized by a stochastic dependence on the system variables. In this instance the initial value problem is

---

* The set of space-time variables will generally be represented as (Z,t); however, the special instances (Z=x,t) and (Z=v,t) will be used in Section III.

always one of choosing the initial probabilities regardless of whether the system variables are ordered or disordered (deterministic or random). Again, this problem is solved by the optimal entropy method.

6. Relation of the Symbolic System to the Real

In order to obtain a physical theory from the above symbolic system, it is necessary to construct one additional decision level within which the relation between the formal symbolic system and the real can be decided. This is generally (although not necessarily) achieved via a mapping (relation) into some number system.

a. Expected Value Formalism

In stochastic theories the correspondence with the real is frequently given in terms of a correspondence between expected values of system variables, or functions thereon, and numbers resulting from measurements on the imaged system.

III. NEURAL NETS AS RELATIONAL SYSTEMS

A. Order Relations in Neural Nets

In the interrogation of nature, the first essential is the extraction and formal symbolization of, at least, a partial system of order relations. For the case of neural nets, it will here be presumed that the kinematic part of the order structure for the neural units is that of a Boolean algebra.(10) That this is a presumption must be underscored since inadequate experience supports the point. For example, it is known that the BA is not the correct order structure for quantum systems, the appropriate structure having only recently been found(5). Moreover, in psychological experimentation on the subjective estimation of probabilities, it is found that subjects are usually conservative estimators relative to formal Bayesian estimation(11). The origin of this conservativism can reside at any of the levels in the relational system hierarchy. In the next two sections (B and C) the conservativism- problem will be discussed in terms of the probabilistic relations and the experiential systemic-subsystemic relations, respectively. With respect to the order structure, let it suffice to point out that a careful analysis of presently presupposed symbolic image systems (e.g., the linguistic structures subsuming propositional systems) is essential.

B. Order-Disorder Relations in Neural Nets

1. Subsystemic Probability Relations

Having tentatively fixed the order structure as a BA, we correspondingly choose the classical form of probability theory as sketched in the previous section. Again, it is necessary, however, to take a further careful look at this problem. This is particularly important in view of the above mentioned observed conservativism in certain behavioral studies(11) which might suggest that the classical experiential probability relations need to be revised.

2. Systemic Probability Relations

In accord with the presumed order relations and subsystemic probability relations for neuronal systems, the systemic probability will be taken as the entropy as discussed above.

C. Experiential System Variables and Space-Time Relations

1. Experimental Background

For the reasons stated in the Introduction, we will confine our discussion to a particular model of neural dynamics,[2,12] namely that developed in the past few years by Cowan. Even though the dynamical equations are derived in an essentially heuristic way, they do incorporate most of what is presently known about the neuron*. In fact, it is important to notice that explicit use is there made of Rall's [13,14] findings concerning the electrotonic spread of post-synaptic potentials from dendrites to soma**. The neural membrane is modeled according to the electric analog introduced by Eccles [15], which takes into account the resting potential, the equilibrium potentials of inhibitory and excitatory post-synaptic potentials, membrane resistance and capacitance, and synaptic conductances. Another major assumption is the introduction of a logistic-like function to relate the mean rate of firing to the mean driving current intensity. While there is at present no evidence to the contrary in the central nervous system of higher animals, it is important to remember that Wachtel and Kandel[16] have found a drastic exception in *Aplysia California* where the same axon may switch from an excitatory effect to an inhibitory one. To complete the experiential background that has to be imaged formally in the dynamical equations, Cowan proceeds then to consider the interactions between neurons. He limits himself to a particular case, namely the reciprocal pathways between some areas of the cerebral cortex and the thalamus in vertebrates***. The essence of these double-way connections are schematically shown in Fig. 1.

---

*This micro-physiological basis is to be compared with the more psychological, nature of other theories a la McCulloch and Pitts, in which the "neuron" is viewed as the elementary unit of behavior and is, sometimes, called a psychon.

** In other words, the all-or-none type of behavior in single central neurons is abandoned. While this "discreteness" is still exhibited by peripheral neurons, especially by those whose function is to carry signals over long distances, central neurons definitely behave in a "graded" fashion. See, for example, the work of Phillips, Shepherd et al. in the olfactory bulb, that led to and confirmed Rall's theory[17].

***From what follows it is clear that, even though it is a particular case, this system is a faithful representation of the "general case", in the sense that, as far as we can see, no important relevant experiential structure is absent therein.

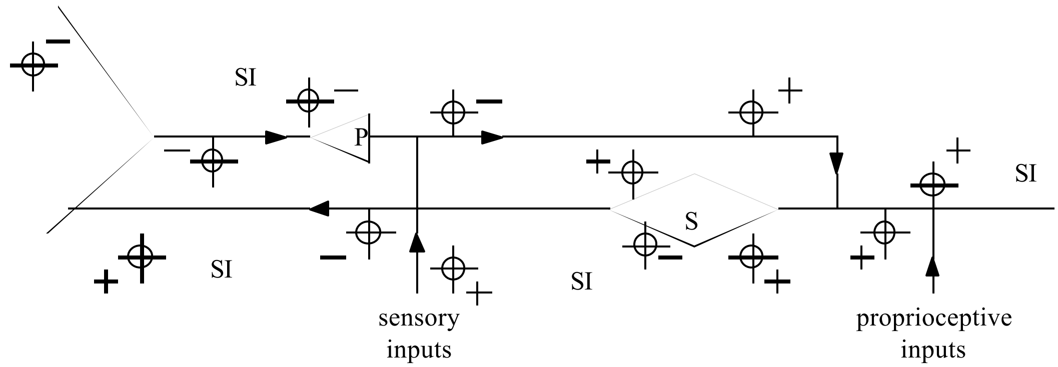

Fig. 1. Corticothalamic circuit. P, pyramidal cells in the cortex; S, stellate cells in the thalamus; SI, stellate interneurons (±, excitatory or inhibitory, respectively). (Redrawn from Cowan [2,12].)

Thalamic stellate cells excite the cortical pyramidal cells, which in turn inhibit the former, etc. Both cells are coupled to a web of interneurons, which are mainly inhibitory with respect to the pyramidal cells, and mainly excitatory with respect to the stellate cells. The interneurons provide a sort of positive feed-back of the activity of the whole net. Cowan has shown that by suitably choosing the parameters that characterize these auxiliary loops, the activity of the circuit will undergo undamped oscillations.

2. Normal Relations

Cowan's general deterministic dynamical equations are then of the following form*

[1] $$x_i(t) = \phi\,[\in_i + (1/\beta_i)\ \sum_j \alpha_{ji} x_j (t - \tau_{ji})].$$

The dynamical variables x_i, called the sensitivities, represent the mean fraction of time during which the neurons are excitable, i.e. not in a refractory state. If δ is the refractory period, and f the mean firing rate, the sensitivity is defined by

$$x(t) = 1 - \delta\ f(t).$$

The dependence on t refers to the long time behavior of the time average. This is a very important point. It is, in fact, not clear whether the actual time patterning of impulses, as opposed to the mean firing rate, carries significant information. It is at this stage that a decision, as to whether, in the initial image of the experiential dynamics, a stochastic picture is to be preferred to a deterministic one, has to be made. For the purpose of the present paper, we accept Cowan's choice, but one has to remember that there is not clear cut evidence in favor of the experientially deterministic approach**. Going back to the fundamental equations, $\in_i$

*Of course, this is a general form. We will consider in a separate paper the question of the most general form compatible with the experimental evidence presently available.
**For example, Siebert[18] has shown that human sound frequency discrimination is in fact consistent with a time-average information processing of the cochlear cell activity. On the other hand, in binaural localization, an interaural difference of a fraction of millisecond can still be detected. *Aplysia* provides another instance of the precise use of time patterns - a train of pulse pairs of constant frequency has different effects depending on the seperation between pairs[19].

represent the *controllable* input to the i-th neuron, while β_i is a kind of conversion factor that measures the relative strength of the neuronal response to the current inputs from the neurons specified in the indexed sum. It is in fact closely related to the threshold. The coupling coefficients α_{ji} incorporate Eccles' electrical model of the somato-dendritic membrane as well as Rall's theory of electrotonic effects. τ_{ji} are the mean intercellular transit times for neural activity. Finally, $\phi(z)$, defined by

$$\phi(z) = e^z/(1 + e^z),$$

is the logistic function.

In their present form, equations [1] are rather intractable. Moreover, as Cowan has shown, they lead to damped oscillations, in the general case. This is in fact a very crucial point. If one does not have a very general procedure for generating a statistical mechanics, a priori not restricted essentially to the equilibrium case, it is necessary to choose a rather narrow dynamical approach, i.e. a hamiltonian formulation. This is why the "naturally" present damped oscillations were transformed away, despite their great interest. Moreover, this is why Cowan was led to approximate his general equations in such a way as to obtain a conserved quantity. The "ad hoc" nature of this procedure has to be confronted with more general and consistent approach outlined in the previous sections. Using the various approximations*, and the coordinate transformation

$$v_i = \ln [(x_i / q_i) / (1 - x_i)],$$

where q_i are the steady state values of the x_i, equations [1] can be rewritten in the following form

$$dv_i / dT = \sum_j (\alpha_{ji} / \beta_j\beta_i) \partial H / \partial v_i .$$

Here

$$H = \sum_i \beta_i [\ln (1 + q_i e^{v_i}) - q_i v_i]$$

and T is a dimensionless parameter defined by

$$T = t / \tau,$$

τ being an average value of the τ_{ji}**. H is thus the conserved quantity (the hamiltonian) upon which Cowan builds his equilibrium stastical mechanics of neural nets. While, as we have repeatedly pointed out, it is altogether unnecessary to constrain the dynamics to exhibit such a conserved quantity in order to apply statistical mechanics, it is very important to realize the heuristic value of such manipulations. There is no guarantee that this way of proceeding will ultimately lead to the prediction of an actually existing macroscopic quantity. At the same time, especially in cases (like the central nervous system) in which it is difficult to decide upon what to look for, this manipulative approach can enable us to make very important educated guesses.

---

*For example[12], that $|d^2\phi(z) / dz^2|$ be sufficiently small. This implies that equations [1] are in fact only valid when $x \to 0$, $x \to 1$, $x \sim .5$.

**Here again the fine structure of the time patterning of neural activity is neglected.

Having established in some way a hamiltonian, it is then a straightforward matter to build a statistical mechanics, and we will not be concerned with the details. We want instead to spend a few words concerning the problem of conserved quantities, since when existent they enter directly in the optimal entropy method. First of all, where are we going to look for such quantities? Second, granted that we have found them, how difficult is it to express them in terms of the dynamical variables? The latter question is indeed relevant. Consider in fact, for example, the question of hormonal action in the central nervous system. It is probably safe to assume that hormones control in some way the activity of particular neural nets, and that in turn neural nets determine the rate of production of particular hormones. If our dynamical equations are expressed in terms of electrical potentials or currents, it is difficult to see how any conserved quantity in terms of hormonal levels could be related to such a formalism. In some sense these considerations give a clue as to the first question. We have to exploit the macroscopic manifestations of the electric microactivity, that is the EEG and similar types of recordings. In this respect the findings of Fox *et al.* [(20,21)] are extremely relevant. They have established an exceptionally high correlation between the probability that a single cell fire and the amplitude of microelectrode EEG (evoked potential). In other words "the general class of brain activity which includes potentials, slow activity, or EEG is representative of the momentary excitability of the brain as indicated by spike probability" [(21)]. With the techniques of multiple microelectrode recordings that are now becoming available[(22),] one can hope to establish spatial correlations between these evoked potentials and possibly the existence of conserved quantities. The latter would then be easily expressed in terms of Cowan dynamical variables.

3. Conservatism in Bayesian Inference

As noted above, the observed conservatism in subjective probability estimation, if not an artifact[(11,22)] may be due to the inconsistency of classical probability theory in imaging this aspect of human inference. In turn this inconsistency may derive from an improper order structure or improper order-disorder relations as indicated previously. Alternatively, however, the probabilistic propositional algebra may be experientially appropriate, whereas the specific experiential systemic-subsystemic relations representing subjective probability estimation may be inadequate. If this latter were the case then the proposed explanations of conservatism[(11)], namely misperception[(24)] or misaggregation of the evidence, would both be expected to obtain. In both instances, however, the formal correction of the model would entail the adjunction of a decision process for forming propositional systems relative to given evidence. In the present relational systems hierarchy this enters only implicitly through the presumed decision structure.

D. Optimal Entropy Inference[(6-9)]

In the preceding Section (III-C) the specific experiential system variables and the space-time relations thereon were formalized for neuronal systems following the development of Cowan. In order to study neuronal networks he has applied the method of classical equilibrium statistical mechanics to collections of neuronal units obeying the specified dynamics. The applicability of this form of statistical mechnaics depends upon whether requirements such as the following are satisfied:

1. equal a priori probabilities;
2. the existence of a conserved quantity;
3. a Liouville theorem;
4. ergodicity.

In the study cited, classical statistical mechanics was only applicable after the dynamics was simplified and transformed so as to satisfy properties 2 and 3.

It is a principal point of this paper that such manipulations, simply for the purpose of applying statistical mechanics, are entirely unnecessary. Indeed, the classical formulations of time-dependent and time-independent statistical mechanics, reversible and irreversible, are deducible as special intsances of the method of optimal entropy inference as introduced above[6-9]. Moreover, OEI, taken as generalized statistical mechanics, is independent of the requirements 1-4 listed above. In this paper we confine our attention to the trivial problem of exhibiting the Cowan model in the OEI formulation. In subsequent work, however, we intend to show the general potential of the OEI method for statistical mechanical studies of neuronal nets, particularly of time-dependent properties.

Since the evidential constraints to be employed are the conserved quantity H and the normalization condition on the distribution function, the form of the OEI processes to be employed is that of the time-independent continuous case. The details of this procedure are as follows:

<u>Given:</u> An exhaustive, mutually exclusive set of propositions corresponding to the possible values of the N-tuple of system variables. In this case the probability function is continuous on the propositions and will be denoted by $\rho_N(V)$ where (V) is the codification of the N system variables (in this instance as real numbers).

<u>Evidence</u>: Also given are the expected value of H

$$\langle H\rangle = \int H(V)\rho_N(V)\,dV,$$

where

$$dV = dv_1\, dv_2 \ldots dv_n,$$

and the normalization condition

$$\int \rho_N(V)\,dV = 1.$$

<u>Problem</u>: To find $\rho_N(V)$.

<u>OEI</u>: The variational problem now becomes

$$\delta[-K\int\rho_N(V)\int\rho_N(V)V - \alpha\int H(V)\,\rho_N(V)\,dV - \lambda_o\int\rho_N(V)\,dV] = 0$$

with the result that

$$\rho_N(V) = \exp\{-[\lambda_o' + \alpha H(V)]\},$$

where $\lambda_o' = \lambda_o + 1$, and the Lagrange multipliers are determined from the normalization condition and the known value of <H>. Applying the normalization condition, this becomes

$$(V) = \frac{H(V)}{\int_{-\infty}^{+\infty} e^{-\alpha H(V)}\,dV} = \frac{H(V)}{\prod_{r=1}^{N}\int_{-\infty}^{+\infty} e^{-\alpha H(V_r)}\,dV_r}$$

It is then shown(2,12) that this reduces to the Beta distribution, i.e. (going back to the original variables)

$$\rho_N = \Pi_r \frac{x_r^{\alpha\beta_r q_r} (1-x_r)^{\alpha\beta_r(1-q_r)}}{B[\alpha\beta_r q_r;\alpha\beta_r(1-q_r)]}$$

where B(u;v) is Euler's Beta Function.

IV. SUMMARY

We have outlined a very general procedure for the foundation and justification of statistical methods in the description of neural nets. For this purpose we consider neural nets as particular cases of relational systems, and proceed to study the minimal hierarchical composition of a physical theory realized within this context. The first level has to do with order relations, namely with the algebra underlying the structure of our representation of neural nets in terms of a set of propositions. We assume this algebra to be Boolean on classical grounds, but we point out that a better justification for such an assumption is needed. The next step consists in the appropriate definition of (subsystemic) probabilities and since we presume a Boolean algebra these are classical probabilities. We then introduce systemic probabilities which include as a special case the entropy function. The next step is to consider the specific experiential system variables and space-time relations and, for the sake of illustration, we use a formalization thereof introduced by Cowan (dynamical equations for the general neuron). The next step is the introduction of a decision structure for the determination of the unknown a priori probability distribution of the dynamic variables. This is the Optimal Entropy Inference Method. Finally the relation of the preceding symbolic system to the real is considered in terms of the relation between expected values of system variables and measurements on the imaged systems.

This general procedure is contrasted with the approach used by Cowan which, by its very nature, has to be confined to equilibrium states. The treatment given here allows for a complete generality, non-equilibrium and non-stationary states can be handled, the only limitations being possibly computational. Since this paper is essentially methodological no specific problem is here considered. We believe however that the present formalism has an important bearing on any theoretical and experimental approach to the problem of neural nets and work is underway-to develop the necessary programs for implementing the method.

ACKNOWLEDGEMENT

We want to express our sincere thanks to Jack Cowan for providing us with some of his unpublished manuscripts on the statistical mechanics of neural nets. Partial support of grants NGR-33-015-016 from NASA, and R01-NB-06682-02 from NIH, is also acknowledged.

REFERENCES

(1) See e.g., Frishkopf, L.S. & Rosenblith, W.A. in Symposium on Information Theory in Biology, H. P. Yockey, R. L. Platzman, H. Quastler, eds., p. 153, Pergamon Press, New York 1958; also see B.D. Burns, The Uncertain Nervous System, Edward Arnold (publishers) Ltd., London 1968.

(2) Cowan, J.D., NRP Session on Neural Coding, Brookline, Mass., 1968 (in press).

(3) Hamann, J.R., Relational Systems: Introduction. An evolving monograph.

(4) Cox, R.T., The Algebra of Probable Inference. Johns Hopkins Univ. Press, Baltimore, Md., 1961.

(5) Piron, C., Helv. Phys. Acta, 37, 439 (1964).

(6) Jaynes, E.T., Phys. Rev., 106, 620 (1957); Phys. Rev., 108, 171 (1957); also see Statistical Physics, K.W. Ford, ed., W. A. Benjamin Inc., New York 1963.

(7) Hamann, J. R., RIAS-TR-66-1, Baltimore, Md. (1966). To appear in Nuovo Cim. (1968).

(8) Lewis, R.M., J. Math. Phys., 8, 1448 (1967).

(9) Katz, A., Principles of Statistical Mechanics: Information Theory Approach, W.H. Freeman & Co., San Francisco, 1967.

(10) McCulloch, W.S. & Pitts, W., Bull. Math. Biophys., 5, 115 (1943).

(11) Edwards, W., in Formal Representation of Human Judgment, B. Kleinmuntz, ed., J. Wiley & Sons, Inc., New York 1968.

(12) Cowan, J.D., Abstract of Lectures given at the 1967 NATO Summer School on Neuronic Networks, Ravello, Italy, 1967 (in press).

(13) Rall, W., Biophys. J., 2, 145 (1962).

(14) Rall, W., in Neural Theory and Modeling, R.F.Reiss, ed. Stanford Univ. Press, Stanford, CA, 1964.

(15) Eccles, J.C., The Physiology of Nerve Cells. Johns Hopkins Univ. Press, Baltimore, Md., 1957.

(16) Wachtel, H. and Kandel, E.R. Science, 158, 1206 (1967).

(17) Rall, W., Shepherd, G.M., Reese, T.S. & Brightman, M.W., Exptl. Neurol., 14, 44 (1966).

(18) Cited in D.M. MacKay, Science, 159, 335 (1968).

(19) Beswick, F.B. & Conroy, R.T.W.L., J. Physiol., (London), 180, 134 (1965).

(20) Fox, S.S., O'Brien, J.H., Science, 147, 888 (1965).

(21) Fox, S.S., Norman, R.J., Science, 159, 1257 (1968).

(22) Verzeano, M., Groves, P., Thomas, S.J., Biophys. J., 8 (Abstract WC1) (1968).

(23) Schum, D.A., Goldstein, I.L., & Southard, J.F., IEEE Trans. on Hum. Fact. in Electronics, 7, 37 (1966).

(24) Beach, L.R., IEEE Trans. on Hum. Fact. in Electronics, 7, 29 (1966).